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Coherent States and N Dimensional Coordinate Noncommutativity

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Abstract: Considering coordinates as operators whose measured values are expectations between generalized coherent states based on the group $SO(N,1)$ leads to coordinate noncommutativity together with full N dimensional rotation invariance. Through the introduction of a gauge potential this theory can additionally be made invariant under N dimensional translations. Fluctuations in coordinate measurements are determined by two scales. For small distances these fluctuations are fixed at the noncommutativity parameter while for larger distances they are proportional to the distance itself divided by a *very* large number. Limits on this number will lbe available from LIGO measurements.

Keywords: Non-Commutative Geometry.

Contents

1. Introduction **1**

2. Coordinates as Coherent State Expectation Values **3**

3. Two Dimensional Noncommutativity Based on Coherent States of the Heisenberg-Weyl Group **5**

4. Two Dimensional Noncommutativity Based on Coherent States of the SO$(2,1) \sim$ SU$(1,1)$ Group **7**

5. Three Dimensional Noncommutativity Based on Coherent States of the SO$(3,1)$ Group **9**

6. Conclusion **11**

A. Coherent States as Solutions of The $SO(2,1)$ Variational Problem **12**

B. Coherent States as Solutions of The $SO(3,1)$ Variational Problem **13**

C. Evaluation of Certain $SO(3,1)$ Matrix Elements **13**

1. Introduction

Most of the recent work on coordinate noncommutativity has restricted itself to two spacial dimensions, where the commutator $[x, y] = i\theta$ is implemented through the Groenewold-Moyal [1] star product; θ is constant for flat two-geometries and somewhat more complicated for spherical ones [2, 3]. In dimensions higher than two this procedure clearly breaks rotational invariance. One of the purposes of this work is to introduce coordinate noncommutativity for unbounded N dimensional spaces and yet maintain $SO(N)$ invariance for the resulting dynamics. In fact, through the introduction of a gauge potential the dynamics can be made invariant invariant under $SO(N, 1)$ transformations. The extra N symmetries, those beyond $SO(N)$, should not be viewed as usual Lorentz transformations as they do not involve time, but rather as "translations". Implementation of such translation symmetries will necessitate the introduction of a gauge potential. The present treatment differs from ones in which Poincaré symmetry is achieved through the introduction of a "twist"

[4, 5] into the action of that algebra while maintaining noncommutativity on a two dimensional plane. The noncommutativity discussed here is fully N dimensional and involves an algebra larger than the one generated by the coordinates. Rotation symmetry is achieved without deforming the angular momentum algebra.

As in ordinary quantum mechanics the coordinates are operators and their measured values are determined by which states matrix elements are taken in. For ordinary, coordinate commuting, quantum mechanics we may choose these states to be simultaneous eigenstates of these coordinates; noncommutativity precludes having a simultaneous eigenstate of all the coordinate operators. Coherent states [6] are those that minimize momentum-position uncertainty and it is tempting to use their analogs for the problem of coordinate noncommutativity. Such states have been applied to investigations of noncommutative geometries [7], principally the ones induced by the star product rules.

One of the primary properties of coherent states is their minimization of the uncertainty in the measurement of noncommuting operators; however none of these measurements have sharp expectation values. In the present formulation we will find that the dispersion of any coordinate introduces two distance regimes.

$$\langle X^2\rangle - \langle X\rangle^2 \sim \bar{\theta} + \frac{1}{\kappa^2}\langle X\rangle^2\,, \tag{1.1}$$

with κ a *very* large number and coordinate noncommutativity determined by $\bar{\theta}$. For distances less than $\kappa\sqrt{\bar{\theta}}$ the fluctuations in measurement of these quantities are of order $\sqrt{\bar{\theta}}$ while for distances greater than $\kappa\sqrt{\bar{\theta}}$ there is a fixed strain of $\sim 1/\kappa$. (The notation $\bar{\theta}$ and its relation to θ will be made apparent in the subsequent text.) As mentioned above, κ has to be very large – how large will be discussed in Section 6.

A general formulation of coordinate noncommutativity based on viewing these coordinates as expectation values of operators between generalized coherent states is presented in Section 2. The procedures for integration and for differentiation of functions of coordinates, when these are treated as expectation values of operators are discussed as is the implementation of translation invariance through the introduction of a gauge potential. The subsequent three sections are specific applications of Section 2 to various dimensions and groups.

Noncommutativity in two spacial dimensions based on Heisenberg-Weyl coherent states is discussed in Section 3. Many modifications of quantum mechanics are the same as those obtained in the star product formulation; it is the presence of fluctuations in the measurement of any length and in the implementation of the translation invariance by the introduction of a gauge potential discussed above that makes the two approaches different. Although the Heisenberg-Weyl group is not in the $SO(N, 1)$ class this section serves as a pedagogical introduction to the use of coherent states in coordinate noncommutativity both in two and in higher number of dimensions.

Two dimensional noncommutativity based on coherent states of the $2+1$ dimensional Lorentz group, $SO(2,1) \sim SU(1,1)$ is discussed in Section 4. Many of the complications that will occur when extending these ideas to higher dimensions will be encountered here, but the algebra is sufficiently simple that results are obtained in closed form. Section 5 goes through in detail the three dimensional case based on $SO(3,1)$ where coordinate noncommutativity and rotational invariance coexist. In Section 6 extensions to dimensions greater than three and the problems of time-space noncommutativity are noted. A discussion of fluctuations in the measurement of any coordinate, as presented in (1.1), is given. Technical details of several calculations may be found in the Appendixes.

2. Coordinates as Coherent State Expectation Values

Our goal is to obtain a theory invariant under N dimensional rotations. To this end we will consider coherent states based on the groups, $SO(N,1)$ [6]; an exception will be for one of two versions of two dimensional noncommutative quantum mechanics, the one based on the Heisenberg-Weyl algebra. We should note that the Heisenberg-Weyl group is a contraction [8] of $SO(2,1)$. As taking over the general discussions of this section to the Heisenberg-Weyl group are straight forward the details will be presented for the $SO(N,1)$ cases. In addition to the $SO(N)$ rotation operators M_{ij}, there are N "noncompact" operators K_i whose commutation relations are

$$[K_i, K_j] = -iM_{ij}\,; \tag{2.1}$$

the minus sign in the above is crucial as it distinguishes this algebra from compact $SO(N+1)$. We will be interested in unitary representations of this algebra that contain all the representations of $SO(N)$ starting with a one dimensional one; the states are labeled as $|j,m;\kappa\rangle$, where $|j,m\rangle$ are irreducible representations of $SO(N)$ and κ^2 is the value of the invariant Casimir operator $K^2 - \frac{1}{2}M_{ij}M_{ij}$; $j = j_0, j_0+1, j_0+2, \ldots$ with the j_0 representation being one dimensional; except for $N=2$, j_0 will be the zero angular momentum state. The usual generalized coherent [6] states are defined as

$$|\vec{\eta};\kappa\rangle = e^{i\vec{\eta}\cdot\vec{K}}|j_0;\kappa\rangle\,. \tag{2.2}$$

The next task is to determine the operators $\vec{X}$ which will represent the coordinates. The measured value of $\vec{X}$ will be $\langle\vec{\eta};\kappa|\vec{X}|\vec{\eta};\kappa\rangle$ and similarly for any observable $F(\vec{X})$ its measured value will be $\langle\vec{\eta};\kappa|F(\vec{X})|\vec{\eta};\kappa\rangle$. This requirement puts several restrictions on the choice of κ and on the choice of the operators $\vec{X}$.

(a) As $|\vec{0};\kappa\rangle$ will be taken to correspond to zero length, $\langle\vec{\eta};\kappa|\vec{X}|\vec{\eta};\kappa\rangle$ should be equal to $\hat{\eta}x(|\eta|)$ with $x(|\eta|)$ monotonic in $|\eta|$ and $x(0) = 0$.

(b) In order to control the fluctuations in the values of the coordinates we require that

$$\langle\vec{\eta};\kappa|X_i^2|\vec{\eta};\kappa\rangle - [\langle\vec{\eta};\kappa|X_i|\vec{\eta};\kappa\rangle]^2 << [\langle\vec{\eta};\kappa|X_i|\vec{\eta};\kappa\rangle]^2 \ , \tag{2.3}$$

or more generally

(c) the cumulants [9] of $\vec{X}$, defined as

$$Q_{ijk\ldots} = \frac{-i\partial}{\partial q_i}\frac{-i\partial}{\partial q_j}\frac{-i\partial}{\partial q_k}\ldots \ln[\langle\vec{\eta};\kappa|e^{i\vec{q}\cdot\vec{X}}|\vec{\eta};\kappa\rangle]|_{q_i=0} \ , \tag{2.4}$$

should satisfy $Q_{ijk\ldots} << Q_iQ_jQ_k\ldots$.

Conditions (b) and (c) require that the states we chose minimize $\langle\vec{\eta};\kappa|\vec{X}^2|\vec{\eta};\kappa\rangle + A\langle\vec{\eta};\kappa|(\vec{X}\cdot\hat{n})^2|\vec{\eta};\kappa\rangle$ subject to the constraint that $\langle\vec{\eta};\kappa|\vec{X}\cdot\hat{n}|\vec{\eta};\kappa\rangle$ is fixed; $\hat{n}$ is the direction along which we wish to measure $\vec{X}$ and the constant A in the above equation allows for a different expectation value for X^2 along $\hat{n}$ and transverse to it. This naturally leads to the variational problem of finding an eigenvector of the operator $(\vec{X})^2 + A(\vec{X}\cdot\vec{n})^2 + \vec{\lambda}\cdot\vec{X}$, with $\vec{\lambda}$ being a Lagrange multiplier fixing the average value of $\vec{X}$. The coherent states, with $\vec{\eta}\sim\vec{\lambda}$ and κ dependent on A, will be shown to be solutions of such an eigenvalue problem; the choice of $\vec{X}$ will also lead to condition (a) being satisfied by these states. Even though the generalized coherent states are solutions of the above discussed variational problem, we shall note that conditions (b) and (c) are satisfied only for representations whose Casimir operators have *very large values*. This condition also prevents large momentum-momentum uncertainty relations. For two spacial dimensions, $\vec{X}$ and $\vec{K}$ will be dual to each other. *This will no longer be the case for higher dimensions*.

A further complication arises for $N\geq 3$. It will be impossible to satisfy conditions (b) and (c) above using expectation values of position operators in coherent states as those discussed till now. Taking a superposition of such matrix elements will solve this problem. Namely we define

$$<< \vec{\eta}|F(\vec{X})|\vec{\eta} >>= \int d\kappa\, h(\kappa)\langle\vec{\eta};\kappa|F(\vec{X})|\vec{\eta};\kappa\rangle \ , \tag{2.5}$$

for a suitably chosen $h(\kappa)$, with $\int d\kappa\, h(\kappa) = 1$; rather then taking expectation values in pure coherent states $|\vec{\eta};\kappa\rangle$ we do it in an ensemble described by the density matrix $\rho = \int d\kappa h(\kappa)|\vec{\eta};\kappa\rangle\langle\vec{\eta};\kappa|$. Expectation values defined in this way *will* satisfy all the above conditions.

An extension of the concepts of integration over space and of differentiation to the present situation in which coordinates are treated as expectation values of operators in coherent states is available. The over completeness [6] of these states and the resolution of unity, $\int \mu(\eta;\kappa)d\vec{\eta}\,|\vec{\eta};\kappa\rangle\langle\vec{\eta};\kappa| = 1$, with $\mu(\eta;\kappa)$ a representation

dependent weight, permits the identifications

$$\int d\vec{x}\, f(\vec{x}) \to N_I \int \mu(\eta;\kappa) d\vec{\eta}\, \langle \vec{\eta};\kappa | f(\vec{X}) | \vec{\eta};\kappa \rangle\,,$$
$$\partial_j f(\vec{x}) \to N_D [iK_j, f(\vec{X})]\,, \tag{2.6}$$

with N_I and N_D constant; N_I is chosen so that, for small η, $N_I \mu(0;\kappa)\, d\vec{\eta} = d\vec{x}$ and N_D is chosen to yield $N_D \langle 0;\kappa | [iK_i, X_j | 0;\kappa\rangle = \delta_{ij}$. Some properties of this "derivative" have to be checked. The over completeness of the coherent states makes $\int \mu(\eta;\kappa) d\vec{\eta}\, \langle \vec{\eta};\kappa | f(\vec{X}) | \vec{\eta};\kappa \rangle$ proportional to $\mathrm{Tr} F(\vec{X})$, where the trace is taken over a specific representation. Thus we find that the integral of a derivative is zero. Likewise, the Leibniz rule is satisfied.

As mentioned earlier, we can extend invariance to the full $SO(N,1)$ group rather than just limiting it to the $SO(N)$ rotations generated by the operators M_{ij}'s. Under the "translations" $\vec{X} \to T^\dagger(\vec{a}) \vec{X} T(\vec{a})$, with $T(\vec{a}) = \exp i\vec{a}\cdot\vec{K}$, completeness of the coherent states and the fact that these are built on a one dimensional representation of the rotation group guarantees

$$\int \mu(\eta;\kappa) d\vec{\eta}\, \langle \vec{\eta};\kappa | T^\dagger(\vec{a}) f(\vec{X}) T(\vec{a}) | \vec{\eta};\kappa \rangle \;=\; \int \mu(\eta;\kappa) d\vec{\eta}\, \langle \vec{\eta};\kappa | f(\vec{X}) | \vec{\eta};\kappa \rangle\,. \tag{2.7}$$

To maintain this invariance for expressions involving derivatives, namely, $[K_j, f(\vec{X})]$ a gauge potential $\mathcal{T}_j(\vec{X})$ has to be introduced with $[iK_j, f(\vec{X})]$ replaced by $[iK_j - i\mathcal{T}_j(\vec{X}), f(\vec{X})]$ and

$$\mathcal{T}_j(\vec{X}) \to T^\dagger(\vec{a}) \mathcal{T}_j(\vec{X}) T(\vec{a}) - T^\dagger(\vec{a}) [K_j, T(\vec{a})]\,. \tag{2.8}$$

The need to modify translations in the context of noncommutative geometries has been discussed in [5]. The field tensor associated with $\mathcal{T}_j$ is

$$\mathcal{F}_{ij} = [K_i, \mathcal{T}_j] - [K_j, \mathcal{T}_i] - [\mathcal{T}_i, \mathcal{T}_j] - [K_i, K_j]\,; \tag{2.9}$$

note that the last term has no analog for the case of ordinary derivatives but is necessary for having $\mathcal{F}_{ij}$ transform under rotations in the expected way, $\mathcal{F}_{ij} \to T^\dagger(\vec{a}) \mathcal{F}_{ij} T(\vec{a})$.

3. Two Dimensional Noncommutativity Based on Coherent States of the Heisenberg-Weyl Group

The Heisenberg-Weyl algebra consists of the elements K_i, $i = 1, 2$ and $\mathbf{1}$ with $[K_i, K_j] = i\epsilon_{ij}\mathbf{1}$. The coordinate operators are taken to be proportional to the dual of the K_i's

$$X_i = \sqrt{\theta}\epsilon_{ij} K_j\,, \tag{3.1}$$

resulting in the commutation relation

$$[X_1, X_2] = i\theta \,. \tag{3.2}$$

We look for a state in which the expectation of $\vec{X}$ is specified and the average of $\vec{X} \cdot \vec{X}$ is a minimum. The standard variational principle leads us to look for an eigenstate of $\vec{X} \cdot \vec{X} - \vec{\lambda} \cdot \vec{X}$, where the λ_i's are Lagrange multipliers. With $|0\rangle$ annihilated by $K_1 + iK_2$, the coherent state

$$|\vec{\eta}\rangle = e^{i\vec{\eta}\cdot\vec{K}}|0\rangle \,, \tag{3.3}$$

with $\vec{\lambda} = 2\sqrt{\theta}\vec{\eta}$ is a solution of the variational equation and

$$\langle\vec{\eta}|\vec{X}|\vec{\eta}\rangle = \sqrt{\theta}\vec{\eta} \,. \tag{3.4}$$

From

$$\langle\vec{\eta}|e^{i\vec{q}\cdot\vec{X}}|\vec{\eta}\rangle = e^{i\sqrt{\theta}\vec{q}\cdot\vec{\eta} - \frac{1}{4}\theta q^2} \tag{3.5}$$

we can find the cumulants of $\vec{X}$, specifically

$$\langle\vec{\eta}|X_i X_j|\vec{\eta}\rangle - \langle\vec{\eta}|X_i|\vec{\eta}\rangle\langle\vec{\eta}|X_j|\vec{\eta}\rangle = \delta_{ij}\frac{\theta}{2} \tag{3.6}$$

and all higher cumulants are zero. As we wish to interpret the expectation values of functions of $\vec{X}$ as measurements of these quantities, it is gratifying that the fluctuations of these position variables are under control. As mentioned in the introduction and in Section 2, this is a guiding principle in choosing states and the operators $\vec{X}$.

Any classical function of the coordinates $f(x) = \int d\vec{q}\, \tilde{f}(\vec{q}) \exp i\vec{q}\cdot\vec{x}$ may be transformed into a similar integral with the c-numbers $\vec{x}$ replaced by the operators $\vec{X}$. The product of exponentials of the $\vec{X}$'s reproduces the star product rules in that

$$e^{i\vec{k}\cdot\vec{X}} e^{i\vec{q}\cdot\vec{X}} = e^{i(\vec{k}\cdot\vec{X} + \vec{q}\cdot\vec{X})} e^{\frac{i\theta}{2}\epsilon_{ij}k_i q_j} \,. \tag{3.7}$$

Following (2.6) we have a transcriptions of integration over space and of differentiation

$$\begin{aligned} \int d\vec{x}\, f(\vec{x}) &\to \frac{\theta}{2\pi}\int d\vec{\eta}\,\langle\vec{\eta}|f(\vec{X})|\eta\rangle \,, \\ \partial_j f(\vec{x}) &\to \frac{-i}{\sqrt{\theta}}[K_j, f(\vec{X})] \,. \end{aligned} \tag{3.8}$$

This also leads to the identification of momentum with the K_i's

$$\vec{p}\psi(\vec{x}) \to \frac{-1}{\sqrt{\theta}}[\vec{K}, \psi(\vec{X})] \tag{3.9}$$

which might lead one to worry that the $\sqrt{\theta}$ in the denominator will introduce a large uncertainty relation for momentum components. This is not the case [10] as we are

making the identification of momenta with commutators of K's and by the Jacobi identities

$$[K_i,[K_j,\psi(\vec{x})]] = [K_j,[K_i,\psi(\vec{x})]\,. \tag{3.10}$$

The observation that

$$[-K_j, e^{i\vec{q}\cdot\vec{X}}] = \sqrt{\theta} q_i e^{i\vec{q}\cdot\vec{X}} \tag{3.11}$$

as an operator identity and (3.7) leads to an algebra identical to the star product one in that

$$\begin{aligned} \frac{\theta}{2\pi}\int d\vec{\eta}\,\langle\vec{\eta}|[K_j,\psi(\vec{X})][K_j,\psi(\vec{X})]|\vec{\eta}\rangle &= -\int d\vec{x}\,\partial_j\psi(x)\partial_j\psi(x)\,,\\ \frac{\theta}{2\pi}\int d\vec{\eta}\,\langle\vec{\eta}|\psi_1(\vec{X})\dots\psi_n(\vec{X})|\vec{\eta}\rangle &= \int d\vec{x}\,\psi_1(x)\star\dots\star\psi_n(x)\,. \end{aligned} \tag{3.12}$$

To see the restrictions of translation invariance it is useful to study two particles interacting by a potential $V(\vec{X}^{(1)}-\vec{X}^{(2)}) = \int d\vec{q}\,\tilde{V}(\vec{q})\exp(i\vec{q}\cdot\vec{X}^{(1)})\exp(-i\vec{q}\cdot\vec{X}^{(2)})$. The potential term in a Lagrangian,

$$\mathcal{L}_V = \frac{\theta^2}{4\pi^2}\int d\vec{\eta}^{(1)}\,d\vec{\eta}^{(2)}\,\langle\vec{\eta}^{(1)},\vec{\eta}^{(2)}|\psi^*(\vec{X}^{(1)},\vec{X}^{(2)})V(\vec{X}^{(1)}-\vec{X}^{(2)})\psi(\vec{X}^{(1)},\vec{X}^{(2)})|\vec{\eta}^{(1)},\vec{\eta}^{(2)}\rangle\,, \tag{3.13}$$

is invariant under the two particle analog of (2.7)

$$\psi(\vec{X}^{(1)},\vec{X}^{(2)}) \to T^\dagger(\vec{a})\psi(\vec{X}^{(1)},\vec{X}^{(2)})T(\vec{a})\,, \tag{3.14}$$

with

$$T(\vec{a}) = e^{i\vec{a}\cdot\vec{K}^{(1)}}e^{i\vec{a}\cdot\vec{K}^{(2)}}\,. \tag{3.15}$$

For the kinetic energy part to be invariant a gauge potential, $\mathcal{T}(\vec{X})$ transforming as in (2.7) has to be introduced.

$$\begin{aligned} \mathcal{L}_K = \frac{-1}{4\pi^2}\int d\vec{\eta}^{(1)}\,d\vec{\eta}^{(2)}\,&\langle\vec{\eta}^{(1)},\vec{\eta}^{(2)}|\\ &\Big\{[K_j^{(1)}-\mathcal{T}_j(\vec{X}^{(1)}),\psi^*(\vec{X}^{(1)},\vec{X}^{(2)})][K_j^{(1)}-\mathcal{T}_j(\vec{X}^{(1)}),\psi(\vec{X}^{(1)},\vec{X}^{(2)})]\\ +\ &[K_j^{(2)}-\mathcal{T}_j(\vec{X}^{(2)}),\psi^*(\vec{X}^{(1)},\vec{X}^{(2)})][K_j^{(2)}-\mathcal{T}_j(\vec{X}^{(2)}),\psi(\vec{X}^{(1)},\vec{X}^{(2)})]\Big\}\,|\vec{\eta}^{(1)},\vec{\eta}^{(2)}\rangle\,; \end{aligned} \tag{3.16}$$

The fact that the values of the coordinates are not sharp and the inclusion of the translational gauge potential $\mathcal{T}(\vec{X})$ differentiates this formalism from the star product one.

4. Two Dimensional Noncommutativity Based on Coherent States of the $\mathbf{SO(2,1)} \sim \mathbf{SU(1,1)}$ Group

Two dimensional coordinate noncommutativity can be obtained using coherent states built on representations of the 2+1 dimensional Lorentz group $SO(2,1)$ whose algebra

is isomorphic to the one for the $SU(1,1)$ group. Three generators J, K_i, with $i = 1, 2$, span this algebra. J is a rotation operator and the K's are boosts. The commutation relations are

$$
\begin{aligned}
[J, K_i] &= i\epsilon_{ij} K_j \\
[K_i, K_j] &= -i\epsilon_{ij} J \,;
\end{aligned}
\tag{4.1}
$$

the minus sign on the right hand side of the lower equation is necessary to distinguish this algebra from the compact $SO(3)$ one. We shall be interested in the unitary, irreducible representations of this group [11] that consist of a tower $|0;j\rangle, |1;j\rangle, |2;j\rangle, \ldots$ of one dimensional representations of J with with eigenvalues greater or equal to zero. With $K_\pm = K_1 \pm K_2$ the action of the operators is

$$
\begin{aligned}
J|n;j\rangle &= (n+j)|j\rangle \\
K_+|n;j\rangle &= \sqrt{(n+1)(n+2j)}\,|n+1;j\rangle \\
K_-|n;j\rangle &= \sqrt{n(n+2j-1)}\,|n-1;j\rangle \,.
\end{aligned}
\tag{4.2}
$$

The Casimir invariant $J^2 - \vec{K}\cdot\vec{K} = j(j-1)$. The coherent states [6, 12] are

$$
|\vec{\eta} : j\rangle = e^{i\vec{\eta}\cdot\vec{K}}|0;j\rangle \,, \tag{4.3}
$$

and the resolution of unity is

$$
1 = \frac{2j-1}{4\pi}\int \sinh\eta\, d\eta\, d\hat{\eta}\, |\vec{\eta};j\rangle\langle\vec{\eta};j| \,. \tag{4.4}
$$

We now have to chose the operator that corresponds to the position vector $\vec{X}$. The simplest choice is, as in the previous section, $X_i = -\sqrt{\theta}\epsilon_{ij}K_j$; the minus sign is for subsequent convenience. (In Appendix A we show that the coherent states are solutions to the appropriate variational problem.) The expectation value of X_i can be obtained from the group algebra,

$$
\begin{aligned}
\langle\vec{\eta};j|\vec{X}|\vec{\eta};j\rangle &= \langle 0;j|\left[\vec{X} - \hat{\eta}\hat{\eta}\cdot\vec{X} + \hat{\eta}(\hat{\eta}\cdot\vec{X}\cosh\eta + \sqrt{\theta}J\sinh\eta)\right]|0;j\rangle \\
&= j\sqrt{\theta}\hat{\eta}\sinh\eta \,.
\end{aligned}
\tag{4.5}
$$

It is immediately obvious that condition (b) in Section 2, eq. (2.3), is *not* satisfied as

$$
\langle\vec{\eta};j|(\hat{\eta}\cdot\vec{X})^2|\vec{\eta};j\rangle - \langle\vec{\eta};j|\hat{\eta}\cdot\vec{X}|\vec{\eta};j\rangle^2 = \frac{j\theta}{2}(\cosh\eta)^2 \tag{4.6}
$$

is of the same order as $\langle\vec{\eta};j|\hat{\eta}\cdot\vec{X}|\vec{\eta};j\rangle^2$. More generally

$$
\langle\vec{\eta};j|e^{i\vec{q}\cdot\vec{X}}|\vec{\eta};j\rangle = \left(\cosh\frac{\sqrt{\theta}q}{2} - i\sinh\frac{\sqrt{\theta}q}{2}\sinh\eta\,\hat{q}\cdot\hat{\eta}\right)^{-2j} , \tag{4.7}
$$

which does not lead to condition (c), eq. (2.4). The desired properties can be recovered in the large j small θ limit, with $j\theta = \bar{\theta}$ fixed. $\bar{\theta}$ sets the noncommutativity scale. To order $1/j$ (4.7) goes over to

$$\langle \vec{\eta}; j | e^{i\vec{q}\cdot\vec{X}} | \vec{\eta}; j \rangle = \exp\left[i \langle \vec{\eta}; j | \vec{q} \cdot \vec{X} | \vec{\eta}; j \rangle - \frac{1}{4} \bar{\theta} q^2 - \frac{1}{4j} \langle \vec{\eta}; j | \vec{q} \cdot \vec{X} | \vec{\eta}; j \rangle^2 \right] . \tag{4.8}$$

To lowest order in $1/j$ we reproduce (3.5) with θ replaced by $\bar{\theta}$ which, following the earlier discussion, shows that in the large j limit $SO(1,2)$ contracts to the Heisenberg-Weyl group. To order $1/j$ (4.6) takes on the form

$$\langle \vec{\eta}; j | (\hat{\eta} \cdot \vec{X})^2 | \vec{\eta}; j \rangle - \langle \vec{\eta}; j | \hat{\eta} \cdot \vec{X} | \vec{\eta}; j \rangle^2 = \frac{1}{2} \bar{\theta} + \frac{1}{2j} \langle \vec{\eta}; j | \hat{\eta} \cdot \vec{X} | \vec{\eta}; j \rangle^2 . \tag{4.9}$$

This has the effect of introducing *two* very disparate distance scales, $\sqrt{\bar{\theta}}$ and $\sqrt{j\bar{\theta}}$. For distances less than $\sqrt{j\bar{\theta}}$ the fluctuations are fixed at the scale of the noncommutativity parameter $\sqrt{\bar{\theta}}$, while for distances on the order of or greater than $\sqrt{j\bar{\theta}}$ the fluctuations grow as the distance itself divided by $\sqrt{j}$. Clearly j has to be very large. We will return to a discussion of this point in Section 6.

The observation that $\langle \vec{\eta}; j | \vec{X} | \vec{\eta}; j \rangle = j\sqrt{\theta} \sinh \eta \, \hat{\eta}$ and (2.6) leads to

$$\begin{aligned} \int d\vec{x} f(\vec{x}) &\to j^2 \theta \int \sinh \eta \, d\eta \, d\hat{\eta} \langle \vec{\eta}; j | f(\vec{X}) | \vec{\eta}; j \rangle , \\ \partial_j F(\vec{x}) &\to \frac{-i}{j\sqrt{\theta}} [K_j, f(\vec{X})] . \end{aligned} \tag{4.10}$$

Again, one has to check whether a large momentum-momentum uncertainty has been introduced. Using Jacobi identities and the algebra of the K_i's and of J,

$$(\partial_i \partial_j - \partial_j \partial_i) \psi(\vec{x}) \to \frac{-i}{j\bar{\theta}} [J, \psi(\vec{X})] . \tag{4.11}$$

As the expectation value of $[J, \psi(\vec{X})]$ will be of the order or less than that of $\psi(\vec{X})$ the momentum-momentum uncertainty will be of the order of $1/(j\bar{\theta})$ which, by the previous argument, will be very small.

5. Three Dimensional Noncommutativity Based on Coherent States of the SO(3, 1) Group

As in the case of $SO(2,1)$ the generators three dimensional Lorentz group, SO(3,1) break up into two classes, the compact ordinary angular momenta J_i's and noncompact K_i's transforming as a vector under angular momentum and with

$$[K_i, K_j] = -i\epsilon_{ijk} J_k . \tag{5.1}$$

The representations of $SO(3,1)$ [13] are made up of towers of the $(2j+1)$ dimensional unitary representations of $SO(3)$, $|j,m\rangle$. We shall be interested in those that start with $j=0$ and thus contain $|0,0;\kappa\rangle, |1,m:\kappa\rangle, |2,m;\kappa\rangle, \ldots$. The value of the Casimir operator $\kappa^2 = K^2 - J^2$ (which we take to be positive) determines the action of the operators K_i on the angular momentum states. The other quadratic Casimir operator $\vec{K}\cdot\vec{J}=0$; in the notation of ref. [13] we are dealing with representations labeled by $l_0 = 0$ and $-l_1^2 = \kappa^2 + 1$.

It is easy to note that choosing $\vec{X}$ to be linear in the generators will not work. From the experience of the previous sections, where a Levi-Civita symbol appeared in the definition of $\vec{X}$, we are led to

$$X_i = \frac{\sqrt{\theta}}{2}\epsilon_{ijk}(J_jK_k + K_kJ_j) \tag{5.2}$$

with

$$[X_i, X_j] = -i\theta\epsilon_{ijk}J_k(K^2+J^2) \tag{5.3}$$

and

$$[K_i, X_j] = -i\sqrt{\theta}\left[\delta_{ij}(K^2+J^2) - K_jK_i - J_jJ_i\right] \tag{5.4}$$

from which, as outlined in previous sections, we can define a derivative.

Let us first look at the expectation values in states of definite κ with $\kappa >> 1$ [14]

$$\begin{aligned}\langle\vec{\eta};\kappa|\hat{\eta}\cdot\vec{X}|\vec{\eta};\kappa\rangle &= \langle 0,0;\kappa|\cosh 2\eta\,\hat{\eta}\cdot\vec{X} \\ &+ \frac{\sqrt{\theta}}{2}\sinh 2\eta\left[\vec{K}^2 - (\hat{\eta}\cdot\vec{K})^2 + \vec{J}^2 - (\hat{\eta}\cdot\vec{J})^2\right]|0,0;\kappa\rangle = \frac{\sqrt{\theta}\kappa^2}{3}\sinh 2\eta\,,\end{aligned} \tag{5.5}$$

while

$$\begin{aligned}\langle\vec{\eta};\kappa|\hat{\eta}\times\vec{X}|\vec{\eta};\kappa\rangle &= \langle 0,0;\kappa|\cosh\eta\,\hat{\eta}\times\vec{X} \\ &- \frac{\sinh\eta}{2}\left(\hat{\eta}\times\vec{K}\hat{\eta}\cdot\vec{K} - \hat{\eta}\cdot\vec{J}\hat{\eta}\times\vec{J}\right)|0,0;\kappa\rangle = 0\,.\end{aligned} \tag{5.6}$$

Details of the proof that these coherent states minimize $\vec{X}^2 + A(\hat{\eta}\cdot\vec{X})^2$, with $\vec{X}$ defined in (5.2), are given in Appendix B.

The leading, in κ^2, contributions to the expectation values of $(\hat{\eta}\cdot\vec{X})^n$ will come from $\langle 0,0;\kappa|(\vec{K}^2 - (\hat{\eta}\cdot\vec{K})^2)^n|0,0;\kappa\rangle$. In order to satisfy conditions (b) an (c) of Section 2 this matrix element has to be equal to $(2\kappa^2/3)^n$. Such a condition was satisfied for the two dimensional case discussed in Section 4 but due to the $(\hat{\eta}\cdot\vec{K})^2$ terms it is not satisfied in the present three dimensional formulation. These matrix elements are evaluated in the Appendix C, (C.6). To leading order in κ

$$\langle\vec{\eta};\kappa|(\hat{\eta}\cdot\vec{X})^n|\vec{\eta};\kappa\rangle = \frac{\sqrt{\pi}}{2}\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\left[\frac{3}{2}\langle\vec{\eta};\kappa|\hat{\eta}\cdot\vec{X}|\vec{\eta};\kappa\rangle\right]^n\,. \tag{5.7}$$

This is unacceptable as it would lead to large fluctuations in the expectation values of the position operator. With the observation

$$\kappa_0^{2n} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\frac{1}{\pi}\int_0^{\kappa_0}\frac{d\kappa}{\sqrt{\kappa_0^2-\kappa^2}}\frac{\partial}{\partial\kappa}\kappa^{2n+1} \tag{5.8}$$

we follow the discussion outlined around (2.5) and consider a new averaging procedure defined by double bras and kets

$$<<\vec{\eta}|\mathcal{O}(\vec{J},\vec{K})|\vec{\eta}>>=\frac{1}{\pi}\int_0^{\kappa_0}\frac{d\kappa}{\sqrt{\kappa_0^2-\kappa^2}}\frac{\partial}{\partial\kappa}\kappa\langle\vec{\eta};\kappa|\mathcal{O}(\vec{J},\vec{K})|\vec{\eta};\kappa\rangle. \tag{5.9}$$

In (5.8) and (5.9) $\kappa_0 >> 1$ replaces κ as a parameter determining the representations of $SO(3,1)$ used. (The lower limit in these integration can be replaced by κ_1 as long as $\kappa_1/\kappa_0 << 1$.)

With this new choice of states we have

$$\begin{aligned} <<\vec{\eta}|\hat{\eta}\cdot\vec{X}|\vec{\eta}>> &= \frac{\sqrt{\theta}\kappa_0^2}{2}\sinh 2\eta\,, \\ <<\vec{\eta}|(\hat{\eta}\cdot\vec{X})^2|\vec{\eta}>> - <<\vec{\eta}|\hat{\eta}\cdot\vec{X}|\vec{\eta}>>^2 &= \theta\kappa_0^2+\frac{4}{\kappa_0^2}<<\vec{\eta}|\hat{\eta}\cdot\vec{X}|\vec{\eta}>>^2\,; \end{aligned} \tag{5.10}$$

As in Section 4 we go to the large κ_0 limit with $\theta\kappa_0^2=\bar{\theta}$ fixed and it is $\bar{\theta}$ that sets the coordinate noncommutativity scale. To order $1/\kappa^2$

$$<<\vec{\eta};j|(\hat{\eta}\cdot\vec{X})^2|\vec{\eta};j>> - <<\vec{\eta};j|\hat{\eta}\cdot\vec{X}|\vec{\eta};j>>^2=\bar{\theta}+\frac{4}{\kappa_0^2}<<\vec{\eta};j|\hat{\eta}\cdot\vec{X}|\vec{\eta};j>>^2\ . \tag{5.11}$$

Again, there are two distance scales. For distances less than $\kappa_0\sqrt{\bar{\theta}}$ the fluctuations are fixed at the noncommutativity parameter $\sqrt{\bar{\theta}}$, while for distances on the order of or greater than $\kappa_0\sqrt{\bar{\theta}}$ the fluctuations grow as the distance itself divided by κ_0. Discussion of these results is left for the next section.

6. Conclusion

We have formulated a coordinate noncommuting quantum mechanics where the measurement of position operators, or functions of such operators, is determined by their expectation values between generalized coherent states. The concepts of integration and of differentiation can be incorporated. This formulation is invariant under the full rotation group and translation invariance holds at the price of introducing a gauge potential. As the measurement of none of the coordinate components is sharp, special attention has to be paid to control any of the fluctuations in the values of these components. This is achieved only for the coherent states built on representations with very large values of the relevant Casimir operators.

Details were presented for two and three dimensions. Extensions to higher dimensions, though algebraically tedious, are straight forward. Coherent states for $SO(N,1)$ groups with $N > 3$ are discussed in [6]. The position operators may be taken as

$$X_i = \frac{\sqrt{\theta}}{2}(M_{ij}K_j + K_jM_{ij}) \,, \tag{6.1}$$

where the M_{ij}'s are the generators of the $SO(N)$ subgroup of $SO(N,1)$. Extension to where time is one of the noncommuting directions is problematical not only for general reasons having to do with violations of unitarity [15] but also due to technical difficulties of extending the present formalism. Naively, to introduce time as one of the noncommuting coordinates one might try to construct coherent states based on the de Sitter group, $SO(N,2)$ with $SO(N,1)$ being the symmetry group and the space-time operators in the coset $SO(N,2)/SO(N,1)$. (It is amusing to note that one of the early attempts at noncommutativity [16] placed the space-time coordinates into such a coset space.) Although such coherent states have, to this authors knowledge, not been studied one problem can be seen immediately: time becomes periodic. This may be noted easily by looking at the previously studied group $SO(1,2)$, now thought of as a de Sitter group.

An intriguing result of this work is displayed in (1.1) where fluctuations in the measurement of any coordinate introduce two distance scales. The first scale is the expected one set by the noncommutativity parameter $\sqrt{\theta}$ while the other one depends on the value of the Casimir operator κ^2 and is equal to $\sqrt{\theta}\kappa$ and induces fluctuations proportional to the length itself. This relation, summarized in (1.1) is reminiscent of the generalized Heisenberg uncertainty relation [17]

$$\Delta x \Delta p \geq \frac{\hbar}{2} + \frac{\theta(\Delta p)^2}{\hbar} \,. \tag{6.2}$$

$\sqrt{\theta}$ is presumably very small, possibly of the order of the Planck length λ_P; κ which acts like a strain will have to be large. Limits of $\kappa > 10^{21}$ will be available from measurements at the LIGO Observatory [18].

A. Coherent States as Solutions of The $SO(2,1)$ Variational Problem

With the choice for coordinate operators made in Section 4 we will show that the coherent states minimize the the expectation of $\vec{X}^2 + A(\hat{\eta}\cdot\vec{X})^2$ with the expectation of $\vec{X}$ fixed. We may take $\hat{\eta}$ along the x direction. We , thus, have to show that for some η and j $|\eta\hat{x};j\rangle$ satisfies the eigenvalue equation

$$[\vec{X}^2 + A(\hat{x}\cdot\vec{X})^2 + \lambda\hat{x}\cdot X]|\eta\hat{x};j\rangle = c|\eta\hat{x};j\rangle \tag{A.1}$$

with λ a Lagrange multiplier and c the eigenvalue. This is equivalent to showing

$$e^{-i\eta K_x}[\vec{X}^2 + A(\hat{x}\cdot\vec{X})^2 - \lambda\hat{x}\cdot X]e^{i\eta K_x}|0;j\rangle = c|0;j\rangle. \tag{A.2}$$

The unitary transformation on the left hand side of the above can be carried out explicitly and using the fact that $K_-|0;j\rangle = 0$ we have to set the coefficients of K_+^2 and K_+ to zero. This leads to the conditions

$$\begin{aligned}\frac{1}{4}\left[(1+A)\cosh^2\eta - 1\right]K_+^2 &= 0\,,\\ (2j\sinh\eta + \lambda)K_+ &= 0\,,\end{aligned} \tag{A.3}$$

or, with $\langle\hat{x}\cdot\vec{X}\rangle = j\sqrt{\theta}\sinh\eta$, we find

$$A = -\frac{\langle\hat{x}\cdot\vec{X}\rangle}{\sqrt{\langle\hat{x}\cdot\vec{X}\rangle^2 + j\bar{\theta}}}\,. \tag{A.4}$$

B. Coherent States as Solutions of The $SO(3,1)$ Variational Problem

We will follow the same procedure for $N=3$ as we did in Appendix A for $N=2$; this time we take $\vec{\eta}$ to be along the $\hat{z}$ direction and show that there are values of η and of κ that lead to a solution of the eigenvalue problem

$$e^{-i\eta K_z}\left[\vec{X}^2 + A(\hat{z}\cdot\vec{X})^2 + \lambda\hat{z}\cdot\vec{X}\right]e^{i\eta K_z}|0,0;\kappa\rangle = c|0,0;\kappa\rangle\,. \tag{B.1}$$

After performing the unitary transformation and noting that $|0,0;\kappa\rangle$ is annihilated by each component of $\vec{J}$ we set the coefficients of K_z^4 and of K_z^2 to zero,

$$\begin{aligned}\sinh^2\eta\left[(1+A)\cosh^2\eta - 1\right]K_z^4 &= 0\,,\\ \left[\left(\cosh^2 2\eta - \frac{\kappa^2}{2}\sinh^2 2\eta\right) - \lambda\sinh 2\eta\right]K_z^2 &= 0\,.\end{aligned} \tag{B.2}$$

Again, with $\langle\hat{z}\cdot\vec{X}\rangle = (\sqrt{\theta}\kappa^2\sinh 2\eta)/3$, A is related to κ,

$$A = \left[1 - \sqrt{1 + \frac{9\langle\hat{z}\cdot\vec{X}\rangle^2}{\kappa^2\bar{\theta}}}\right]\left[1 + \sqrt{1 + \frac{9\langle\hat{z}\cdot\vec{X}\rangle^2}{\kappa^2\bar{\theta}}}\right]^{-1} \tag{B.3}$$

C. Evaluation of Certain $SO(3,1)$ Matrix Elements

The matrix element $\langle 0,0;\kappa|\left[\vec{K}^2 - (\hat{\eta}\cdot\vec{K})^2\right]^n|0,0;\kappa\rangle$, for $n < \kappa$, needed in Section 5 will be evaluated. By rotational invariance we may set $\hat{\eta} = \hat{z}$ and evaluate

$\langle 0,0;\kappa|(\vec{K}^2 - K_z^2)^n|0,0;\kappa\rangle$. The states $|l,0;\kappa\rangle$ are eigenstates of $\vec{K}^2$ with eigenvalue $\kappa^2 + l(l+1) \sim \kappa^2$. In order to obtain $\langle 0,0;\kappa|K_z^{2\mu}|0,0;\kappa\rangle$, with μ integer we evaluate the coefficients $\alpha_l^{(2\mu)}$ in the expansion

$$K_z^\mu|0,0;\kappa\rangle = \kappa^\mu \sum_l (-i)^l \sqrt{2l+1}\alpha_l^{(\mu)}|l,0;\kappa\rangle\,; \tag{C.1}$$

l is restricted to even values. For the representation of interest and for $\kappa >> l$ the action of K_z on the states $|l,0;\kappa\rangle$ is [13]

$$K_z|l,0;\kappa\rangle = i\kappa\left(\frac{l}{\sqrt{4l^2-1}}|l-1,0;\kappa\rangle - \frac{l+1}{\sqrt{4(l+1)^2+1}}|l+1,0;\kappa\rangle\right)\,. \tag{C.2}$$

It is straightforward to obtain the recursion relation for the $\alpha_l^{(\mu)}$'s ($\alpha_0^{(0)} = 1$),

$$(2l+1)\alpha_l^{(\mu+1)} = (l+1)\alpha_{l+1}^{(\mu)} + l\alpha_{l-1}^{(\mu)}\,, \tag{C.3}$$

whose solution is

$$\alpha_l^{(\mu)} = \frac{1}{2}\frac{(\frac{\mu}{2})!(\frac{\mu-1}{2})!}{(\frac{\mu-l}{2})!(\frac{\mu+l+1}{2})!} \tag{C.4}$$

and especially

$$\alpha_0^{(\mu)} = \frac{1}{\mu+1}\,. \tag{C.5}$$

From the above we obtain

$$\begin{aligned}\langle 0,0;\kappa|(\vec{K}^2 - K_z^2)^n|0,0;\kappa\rangle &= \kappa^{2n}\sum_\mu (-1)^\mu \frac{n!}{\mu!(n-\mu)!}\frac{1}{2\mu+1} \\ &= \kappa^{2n}\frac{\sqrt{\pi}}{2}\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\end{aligned} \tag{C.6}$$

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